JACOBIAN TRANSFORMATION IN FINITE ELEMENTS METHOD

A preliminary knowledge related to partial differential equations is given:

Let's consider a function which depends on the variables k and n. Moreover, k and n depend on the variable t so that

$$f = f(k,n) \quad k = k(t) \quad n = n(t)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial k}\frac{dk}{dt} + \frac{\partial f}{\partial n}\frac{dn}{dt}$$

Multiply both sides with dt:

$$df = \frac{\partial f}{\partial k}dk + \frac{\partial f}{\partial n}dn$$

Now, define the coordinate field in terms of bilinear shape functions N_i:

$$x = \sum N_i\, x_i$$

$$y = \sum N_i\, y_i$$

where $N = N(\zeta,\eta)$, so $x = x(\zeta,\eta)$ and $y = y(\zeta,\eta)$

Writing in differential form:

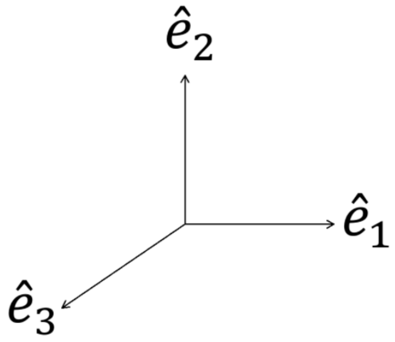
$$dx = \frac{\partial x}{\partial \zeta}d\zeta + \frac{\partial x}{\partial \eta}d\eta$$

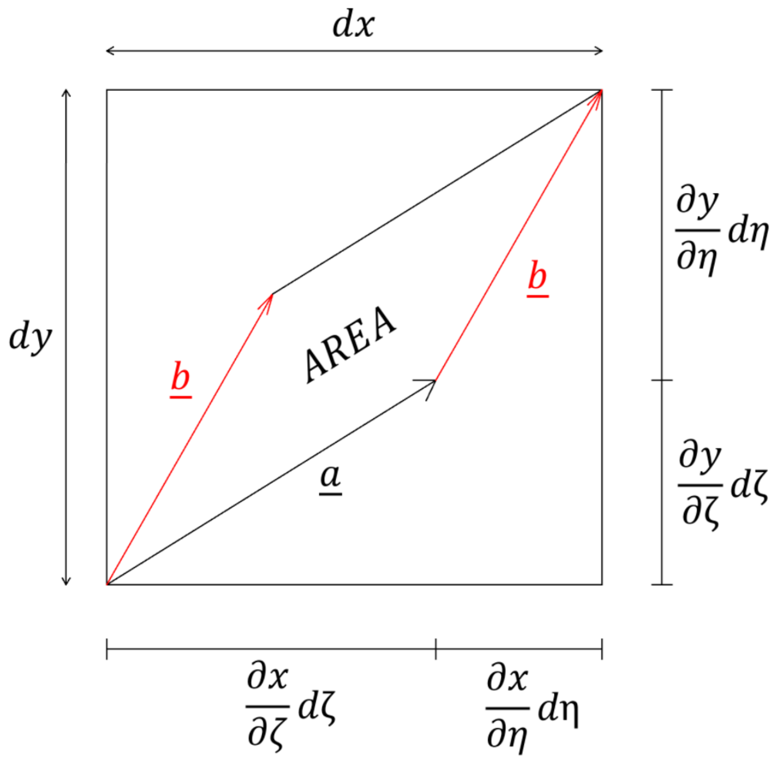
$$dy = \frac{\partial y}{\partial \zeta}d\zeta + \frac{\partial y}{\partial \eta}d\eta$$

We can visualise this components with vectors as seen in the figure below. The vectors $\underline{a}$ and $\underline{b}$ shows the components in the direction of ζ and η. These are the basis of the natural coordinate system transformed from cartesian system with x and y.

The vectors $\underline{a}$ and $\underline{b}$:

$$\underline{a} = \left(\frac{\partial x}{\partial \zeta}, \frac{\partial y}{\partial \zeta}\right)d\zeta \qquad \underline{b} = \left(\frac{\partial x}{\partial \eta}, \frac{\partial y}{\partial \eta}\right)d\eta$$





Since the vectors $\underline{a}$ *and* $\underline{b}$ *are in the directions of* ζ *and* η*, the area:*

$$A = \underline{a} \, x \, \underline{b}$$

$$= \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & 0 \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & 0 \end{vmatrix} d\zeta \, d\eta$$

$$= \begin{vmatrix} \frac{\partial x}{\partial \zeta} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \zeta} & \frac{\partial y}{\partial \eta} \end{vmatrix} d\zeta \, d\eta \, \hat{e}_3$$

$$\boxed{dA = |J| \, d\zeta \, d\eta}$$

The area defined by x and y basis (rectangular area):

$$\boxed{dA = dx \, dy}$$

Finally:

$$\boxed{dx \, dy = |J| \, d\zeta \, d\eta}$$